

For what number of cars must self organization occur in the Biham-Middleton-Levine traffic model from any possible starting configuration?

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Abstract

For any initial configuration of fewer than $\frac{1}{2}N$ cars the BML model will self organize to attain speed one. On the other hand, there is a configuration of size m in which no car can move if and only if m is at least $2N$.

Write $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})$, and consider the $N \times N$ discrete torus $\mathbb{Z}_N \times \mathbb{Z}_N$. We think of the first coordinate as pointing vertically and the second as pointing horizontally. Suppose we choose $m < N^2$ points of this torus at random, and then assign to each of them a colour – red or blue – independently with equal probabilities. We think of these points as little cars on the torus. Having once been positioned, the cars move at discrete time-steps according to the following rules:

1. Red cars try to move one step upwards, and blue cars try to move one step to the right.
2. However, two cars may not occupy the same space. In any given time step, first all the blue cars try to move, then all the red cars. If, when the time comes to move, the space immediately to the right of a blue car is already occupied by a red car, then the blue car – and any more blue cars in a line to the left of it – cannot move, so stay still. Then, if any vertical line of red cars is blocked by a blue car above them, they all stay still.

In the classical Biham-Middleton-Levine (BML) model ([2]), the number m is taken to be pN^2 for some $p \in (0, 1)$. Numerical simulations suggest that the system exhibits a phase transition: there is a certain critical value of p , say p_c (thought to be roughly 0.3), such that for $p > p_c$, asymptotically almost surely every car is blocked from moving by some other car after finitely many steps (we say the system is **stuck**), whereas for $p < p_c$, asymptotically almost surely the system self-organizes: every car is blocked from moving only finitely many times, and so after some finite time the cars all move freely (we say the system attains **speed one**).

This behaviour has been witnessed in a number of computer simulations for increasingly large N . However, recently a more subtle picture has started to emerge: intermediate phases of behaviour have been observed between attaining speed one and getting stuck, and it also now appears possible that there is no $p_c \in (0, 1)$, but rather that as $N \rightarrow \infty$ the behaviour changes increasingly sharply around a point $p_c(N)$ that decays very slowly to 0. See, in particular, D’Souza [3]. However, to date the only result to be proved rigorously (by Angel, Holroyd and Martin [1]) is that the system gets stuck for all p sufficiently large. Furthermore, when the implicit lower bound for such p in this last result is computed it is found to be very close to 1.

In this work we consider a related deterministic question, which turns out to be much simpler: Given N , for which m does *any* initial configuration of the cars inevitably attain speed one, and for which is there some initial configuration of the cars which gets stuck?

Proposition 1 *If $m < \frac{1}{2}N$, the system must attain speed one.*

Remark Of course, this bound is very much lower than the threshold ($\sim pN^2$) originally observed for the random model.

Proof The idea here is to consider the locations of the cars on the N NW-SE diagonals of the torus. Let these be D_1, D_2, \dots, D_N :

$$D_k = \{(i, j) \in \mathbb{Z}_N^2 : i + j = k \pmod{N}\}.$$

If a car moves during a time step, then it moves up one diagonal; else it stays still. For each $t \geq 0$ let $\phi_t : \mathbb{Z}_N^2 \rightarrow \mathbb{Z}_N$ be the associated ‘time-corrected diagonal map’: $\phi(i, j) = i + j - t \pmod{N}$.

Suppose X^1, X^2, \dots, X^m are the initial positions of our cars, and write X_t^i for the position of car i at time $t \geq 0$ (so $X_0^i = X^i$) and $Y_t^i = \phi_t(X_t^i)$. Thus, knowing the Y_t^i at a given time t tells us something about the configuration $(X_t^i)_{i \leq m}$, but far from specifying it uniquely. Our proof will use constraints on the behaviour of $Y_t^1, Y_t^2, \dots, Y_t^m$ as t increases.

At a given time t , the points Y_t^i are distributed within \mathbb{Z}_N : some points of \mathbb{Z}_N may be occupied by many such Y_t^i (several cars may occupy different points on the same diagonal), while others will be empty. We will partition the set $\mathbb{Z}_N \setminus \{Y_t^i : i \leq m\}$ of empty points at time t into a union of arcs in \mathbb{Z}_N , say $A_t^1 \cup A_t^2 \cup \dots \cup A_t^{r(t)}$, where (for sake of argument) we label the arcs in order and choose A_t^1 to be the arc containing the first non-occupied point when \mathbb{Z}_N is written as $\{0, 1, \dots, N\}$.

Lemma 2 *Suppose that at time t some arc $A_t^s = \{y, y + 1, \dots, y + l\}$ has length at least 2. Then at time $t + 1$ the point immediately to its left, $y - 1$, is still occupied by some Y_{t+1}^i , and the set $A_t^s \setminus \{y + l\} = \{y, y + 1, \dots, y + l - 1\}$ is still an arc of unoccupied points.*

Proof of Lemma 2 This is a direct observation from the dynamics of the cars. The lower boundary point cannot move, since given a number of cars all in the same diagonal in the discrete torus with no cars in the diagonal above, at least one of those cars will not be blocked during the next time step; so, under the time-corrected diagonal map, at least one of the images of those cars must stay still in this time step. Also, images of cars under the maps ϕ can only either stay still or move one step to the left in one time step, so it is clear that the points $\{y, y + 1, \dots, y + l - 1\}$ cannot become occupied during the next time step. \square

Lemma 3 *The dynamics cannot create new arcs A_t^s of length greater than 1: the number of such long arcs is non-increasing in t .*

Proof of Lemma 3 Suppose that at time $t + 1$ we have an arc A_{t+1}^s of length at least 2. During the time step from t to $t + 1$, the images of those cars that are now in diagonals immediately above and below A_{t+1}^s either stayed still or moved one step to the left. Thus, by Lemma 2, at time t there must have been an empty arc at least as long as A_{t+1}^s and with the same lower end-point. Thus to each empty arc of length at least 2 at time $t + 1$ we can associate such an arc at time t ; since this association is also clearly unique, the number of such arcs cannot increase. \square

Lemma 4 *If the system never attains speed one, then there must come a time when no arcs A_t^s have length greater than 1.*

Proof of Lemma 4 The point is that if there are infinitely many times at which cars are blocked, then in particular some car must be blocked infinitely often. Suppose it is car $i \leq m$. This means that as t increases Y_t^i describes infinitely many circuits around the discrete circle \mathbb{Z}_N .

But after completing one such circuit (say at time T), there can be no arcs of length greater than 1 remaining; for if there is still an arc A_T^s of length at least 2 at time T , then, arguing as in the proof of the previous lemma, there must be a sequence of arcs $A_t^{s(t)}$ for $t = 0, 1, \dots, T$, all of length at least 2, and all with the same lower end-point, say y . Thus we deduce that $y \in \mathbb{Z}_N$ must remain occupied, and have an arc of length at least 2 immediately to its right, for all times $t \leq T$. This contradicts the observation that Y^i passes through point of \mathbb{Z}_N by time T . \square

Completion of proof of Proposition 1 This is now immediate: if $m < \frac{1}{2}N$ then, however the images $Y_t^1, Y_t^2, \dots, Y_t^m$ are distributed in \mathbb{Z}_N , there will always be some empty arc of length at least 2, and so, by Lemma 4, the system must attain speed one in finite time. \square

Thus the system must self-organize from any (deterministic) starting configuration with fewer than $\frac{1}{2}N$ cars. A related randomized question is:

Question 5 *Suppose we place a configuration of $m < \frac{1}{2}N$ cars on $\mathbb{Z}_N \times \mathbb{Z}_N$ uniformly at random. By Proposition 1 the system will self organize to attain speed one, but how many collisions will occur before it does so?*

The results above notwithstanding, problems can arise with as few as $2N$ cars:

Proposition 6 *There is a configuration with m cars which is stuck if and only if $m \geq 2N$.*

Proof (\Rightarrow) Observe first that no column can contain some red but no blue cars and no row can contain some blue but no red cars, as then those cars would be able to move freely. Now suppose the system is stuck. Then, in particular, in every column there must be at least one blue car: for there cannot be only red cars, and if there were no cars in that column, then there would be blue cars in some column to the left of it which are not blocked. Similarly, in every row there must be at least one red car. Thus, there are at least N red cars and at least N blue cars, so $m \geq 2N$.

(\Leftarrow) We need only witness a configuration of $2N$ stuck cars: choose two adjacent SW-NE diagonals, and occupy the lower entirely with red cars and the upper entirely with blue cars. Now for any $m \geq 2N$, we can add more cars that are also blocked to this configuration to deduce the result. \square

This leaves the following question for the deterministic problem:

Question 7 *Does the system necessarily self-organize to attain speed 1 for any $m \geq \frac{1}{2}N$? Put differently, for which $\frac{1}{2}N \leq m < 2N$ can the system stay forever below speed 1, even though (by Proposition 6) it can never get stuck?*

We remark that the idea underlying Proposition 6 can also be brought to bear on a version of the model with a (fairly sparse) random initial configuration:

Proposition 8 *If a configuration of $n \log n$ red cars and $n \log n$ blue cars is selected in $\mathbb{Z}_N \times \mathbb{Z}_N$ uniformly at random, then asymptotically almost surely the system never becomes stuck.*

Proof We use again the observation that any stuck configuration must contain at least one blue car in every column and at least one red car in every row. However, with $n \log n$ cars of each colour, asymptotically almost surely there is some column with no blue cars or some row with no red cars; this gives the result. \square

Finally, we see no harm in asking:

Question 9 *Could ideas such as those above be used to show that in the original (random) BML model self organization occurs (asymptotically almost surely) for $N^{1+\alpha}$ cars for some $0 < \alpha < 1$?*

References

- [1] O. Angel, A.E. Holroyd & J.B. Martin, “The jammed phase of the Biham-Middleton-Levine traffic model.”, *Elect. Comm. in Probab.* 10 (2005), 167 – 178.
- [2] O. Biham, A.A. Middleton & D. Levine, “Self-organization and a dynamical transition in traffic flow models.”, *Phys. Rev. A* 46 (1992), R6124.
- [3] R.M. D’Souza, “Geometric structure of coexisting phases found in the Biham-Middleton-Levine traffic model.”, *Phys. Rev. E* 71 (2005), 0066112.